HAT PROBLEMS ON BIPARTITE GRAPHS AND LINE OF SAGES PROBLEMS

WILLIAM BASS, TYLER HUANG, REYNALD OLIVERIA, KEVIN A. ZHOU

Abstract. Hat problems are a staple in recreational mathematics. Usually these problems involve people with hats and they can see everyone else's hats but theirs, or participant 1 only seeing participant 2's hat, participant 2 seeing only participant 3's hat and so on, or people stand in a circle and can only see the hats of the people standing immediately next to them. These three situations can be represented by a graph that we call a sight graph. For example, the first case above would be a complete graph. Many papers prior have explored hat problems with differing sight graphs and this continues such tradition. We prove that for a sight graph of $K_{m,n}$ and three hat colors, then the number of guaranteed answers is $\left\lfloor \frac{\min(m,n)}{2} \right\rfloor$. A similar problem is the Line of Sages problem, presented by Tanya Khovanova [1]. We explore the generalized version of varying number of colors and people and we have shown that for any number of colors and three people, there is a strategy to guarantee two correct answers.

1. Introduction

In general, hat problems involve people who are wearing hats of different colors but are not able to see their own hat. In one way or another, each participant will try to guess their own hat color based on some information given. The variety comes in the goals and knowledge of each participant. There are hat problems where the people wearing hats attempt to maximize the correct answers *most* of the time. Others require that no one participant will answer incorrectly given that they are able to pass. By nature, some hat problems require probabilistic strategies, whereas some require deterministic strategies. The website, https://www.cs.umd.edu/ gasarch/TOPICS/hats/hats.html, is a compilation of various papers on hat problems.

There are two types of hat problems that this paper focuses on. First are a set of hat problems where the people who are wearing hats can see other people's hats but cannot see their own. Which participant can see who can vary. The sight of each participant can be represented by a graph whose vertices represent each participant and an edge shows that the two people can see each other. This is called a sight graph. There has been some work on hat problems with a variety of sight graphs like in [2]. Our work continues this tradition. Finally, we generalize of the Line of Sages problem, proposed by Tanya Khovanova [1].

Contents

2. Hat Problems on Bipartite Graphs

2.1. **Setting.** The general type of hat problem considered in this section is one where n participants are wearing hats. Each participant cannot see their own hat. Then, some adversary then paints each hat one of k colors. This adversary will be referred to as the mad hatter. After the hats are painted, the mad hatter then asks the participants to all simultaneously "guess" the color of the hat they are wearing. And the group of participants will earn points corresponding to the number of correct guesses. The problem is then: Which strategy can guarantee the most amount of points?

A keyword in the problem statement is "guarantee" which means that any strategy that involves a random guess is automatically ruled out, as such a strategy does not guarantee anything. Thus, for an individual participant, a strategy dictates exactly how they guess based solely on the colors of the hats of the other participants. That is, given a strategy, an individual participant's "guess" is determined exactly by the colors of the hats of the other participants. In a sense, the word "guess" is not used in the common way here. The strategy for the whole group, then, is a collection of these individual strategies.

This problem may be varied by changing the number of participants n, and the number of colors k. But, the mad hatter may also change which participants each participant gets to see. For example, suppose the participants are Ada, Bob, Charlie, and Meir. The mad hatter may permit each participant to see every other participant, or the mad hatter may split the group into pairs and let the pairs only see each other. In the perspective of Alice, the first case corresponds to Alice being able to see Bob, Charlie, and Meir. The second case corresponds to Alice and Bob being able to see only each other, and Charlie and Meir being able to see only each other. These are not the only possible cases for these 4 participants. Who can see whom can be denoted using a graph, now aptly named a sight graph:

(A) First Case (B) Second Case

FIGURE 1. Example Sight Graphs

A sight graph in itself implies the number of participants. And so, a hat problem, as we will consider, is completely defined by a sight graph G, and a number of colors k. Strategies are then created under this context. And so the problem generalizes to: Given a sight graph G, and a number of hat colors k, which strategy can guarantee the most amount of points? Or less specifically, what is the most points can a strategy guarantee?

2.2. **Notation.** Notation can be introduced to formalize the above setting.

Notation 2.2.1. *A hat problem is a tuple (G, k) where $G = (V, E)$ is a simple, undirected graph, and k is a positive integer. We assume that if G has n vertices, then $V = \{1, 2, \ldots, n\}$.*

Consider a hat problem $(G = (V, E), k)$. V can be interpreted as the set of participants wearing hats. E lists which participants can see each other, and k is the number of different hat colors. The rest of the notation is defined within the context of a fixed hat problem $(G = (V, E), k)$, and we let $n = |V|$.

Notation 2.2.2. *A configuration, c, is a function from V to $\mathbb{Z}_k$. With the assumption that $V = \{1, 2, \ldots, n\}$, equivalently, a configuration c is a member of $\mathbb{Z}_k^n$. Further, $\mathscr{C}$ is the set of all configurations.*

A configuration c can be interpreted as one possible way in which the mad hatter paints the hats of the participant. As a function, it takes in a participant $v \in V$, and outputs the color of their hat ($i \in \mathbb{Z}_k$) which is one of k possible colors.

Notation 2.2.3. *An individual strategy for a $v \in V$, s_v, is a function from $\mathbb{Z}_k^{\deg(v)}$ to $\mathbb{Z}_k$. A strategy for the whole group, or simply a strategy, is a tuple of strategies $\bar{s} = (s_1, s_2, \ldots, s_n) = (s_v : v \in V)$. $\mathcal{S}$ is the set of all strategies.*

An individual strategy dictates what an individual participant guesses based on the hat colors of the participant's neighbors in the sight graph. The input for an individual strategy takes in a tuple of colors. This tuple of colors dictates the hat colors of the participant's neighbors arranged from least to greatest, as we assume that $V = \{1, 2, \ldots, n\}$. The output of an individual strategy is a color, which is interpreted as the guess of the participant.

Definition 2.2.4. *Consider a $v \in V$. Let $\{v_1, v_2, \ldots, v_m\}$, where $v_1 < v_2 < \cdots v_m$, be the set of vertices adjacent to v in G. Further, let $\bar{s} = (s_w : w \in V)$ be a strategy. For any configuration $c \in \mathscr{C}$, define $s_v\{c\} = s_v(c(v_1), c(v_2), \ldots, c(v_m))$ and define $\bar{s}\{c\} = (s_w\{c\} : w \in V)$. Next, define $s_v[c] = 1$ if $s_v\{c\} = c(v)$ and $s_v[c] = 0$ otherwise. Further define, $\bar{s}[c] = \sum_{v \in V} s_v[c]$. Finally define:*

$$\bar{s}[\mathscr{C}] = \min_{c \in \mathscr{C}} \bar{s}[c]$$

Given a strategy $\bar{s}$ and a configuration c, the above definition notates $s_v\{c\}$ in a way that can be interpreted as the guess of participant v under the hat color configuration c. Extending this, $\bar{s}\{c\}$ is a tuple representing the guesses of each participant under the configuration c. Further, the definition also gives $s_v[c]$ which can be interpreted as being 1 when participant v guesses correctly, and 0 when participant v guesses incorrectly. And so, $\bar{s}[c]$ is the number of correct guesses the strategy $\bar{s}$ yields given configuration c. Then, $\bar{s}[\mathscr{C}]$ is the number of correct guesses that the strategy $\bar{s}$ guarantees. With this notation, we can define more precisely the problem stated above.

Definition 2.2.5. *Given a hat problem* (G, k)*, then* $\mathrm{HAT}(G, k)$ *is defined as:*

$$\mathrm{HAT}(G, k) := \max_{\bar{s} \in \mathcal{S}} \min_{c \in \mathscr{C}} \bar{s}[c] = \max_{\bar{s} \in \mathcal{S}} \bar{s}[\mathscr{C}]$$

And so, $\mathrm{HAT}(G, k)$ is the answer to the question: given the sight graph G, and k hat colors, what is the most points can a strategy guarantee?

2.3. **Preliminary Results.** Many values of HAT is known. In Butler et. al [2] alone, we find that $\mathrm{HAT}(K_m, k) = \lfloor m/k \rfloor$, where K_m is the complete graph of m vertices, $\mathrm{HAT}(G, 2)$ is the maximal matching of G (that is the maximum amount of 2-cliques in G), and that $\mathrm{HAT}(T, k) = 0$ for $k \geq 3$, where T is a tree graph. A principle implicitly used by Butler et. al is that if a graph G contains independent subgraphs $G_1, G_2, \ldots, G_m$, then:

$$\mathrm{HAT}(G, k) \geq \mathrm{HAT}(G_1, k) + \mathrm{HAT}(G_2, k) + \cdots + \mathrm{HAT}(G_m, k)$$

The proof of this principle is notationally heavy but conceptually intuitive. Informally put, the participants in G_i can perform the strategy that yields $\mathrm{HAT}(G_i, k)$ for $1 \leq i \leq m$, independent of the other participants in the other subgraphs. Such a strategy would then yield $\mathrm{HAT}(G_1, k) + \cdots + \mathrm{HAT}(G_m, k)$, and so, $\mathrm{HAT}(G, k)$ must be at least that sum.

Beyond this principle, the key result used for the bipartite case is one found by Witold Szczechla [3] namely:

Theorem 2.3.1. *Denote the cycle graph of* n *vertices to be* C_n*.* $\mathrm{HAT}(C_n, 3) = 1$ *if* n *is divisible by* 3 *or* $n = 4$*. Otherwise,* $\mathrm{HAT}(C_n, 3) = 0$*.*

2.4. **Primary Result.** In keeping with standard notation, we denote $K_{m,n} = (V = V_m \sqcup V_n, E)$ to be a complete bipartite graph with m being the size of one bipartition, and n the size of the other. And, we assume without loss of generality, $V = \{1, 2, \ldots, m, m{+}1, \ldots, m{+}n\}$. Finally, we also assume that $V_m = \{1, 2, \ldots, m\}$ is one bipartition, and $V_n = \{m+1, m+2, \ldots, m+n\}$ is the other.

Theorem 2.4.1. $\mathrm{HAT}(K_{m,n}, 3) = \left\lfloor \dfrac{\min(m, n)}{2} \right\rfloor$

Proof. Without loss of generality, let $m \leq n$. First, note that taking two vertices from V_m and two from V_n, as well as the edges between those vertices, yields a subgraph isomorphic to the 4-cycle. And since $m \leq n$, we can find at least $\lfloor m/2 \rfloor$ independent 4-cyclic subgraphs. Combining the previous theorem, and the aforementioned principle, we see that $\mathrm{HAT}(K_{m,n}, 3) \geq \lfloor m/2 \rfloor$.

And thus, it suffices to prove that $\mathrm{HAT}(K_{m,n}, 3) \leq \lfloor m/2 \rfloor$. And so, consider an arbitrary strategy $\bar{s} = (s_1, \ldots, s_{m+n})$. Let c_0 be any configuration in which $c_0(i) = 0$ for all $i \leq m$. Note that, because the graph is bipartite, $s_j\{c_0\}$, for

$j > m$, does not change for any choice of such c_0. Less formally, c_0 is an arbitrary configuration in which all participants in V_m wears hat color 0. Thus, the guesses of the participants in V_n is fixed for any such configuration.

In a similar manner, let c_1 be any configuration in which $c_1(i) = 1$ for all $i \leq m$. And we can again conclude that $s_j\{c_1\}$ does not change for any such choice of c_1 for $j > m$. Now, for any $j > m$, there is a member of $\mathbb{Z}_3$ which is not in $\{s_j\{c_0\}, s_j\{c_1\}\}$. Let such a member be a_j. Now, define c'_0 and c'_1 as follows, $c'_0(j) = c'_1(j) = a_j$ for $j > m$, and $c'_0(i) = 0$, $c'_1(i) = 1$ for $i \leq m$. And thus, by construction, $s_j[c'_0] = s_j[c'_1] = 0$. Therefore, $\bar{s}[c'_0]$ is the number of $i \leq m$ such that $s_i[c'_0] = 1$, or equivalently $s_i\{c'_0\} = 0$. Likewise, $\bar{s}[c'_1]$ is the number of $i \leq m$ such that $s_i[c'_1] = 1$, or equivalently $s_i\{c'_1\} = 1$.

Furthermore, since $c'_0(j) = c'_1(j)$ for $j > m$, then $s_i\{c'_0\} = s_i\{c'_1\}$. Again, less formally, since the participants in V_n wear the same color of hats in c'_0 and c'_1, then the guesses of the participants in V_m are the same for both configurations. Suppose there are at least $\lfloor m/2 \rfloor + 1$ of $i \in \{1, \dots, m\}$ such that $s_i\{c'_0\} = s_i\{c'_1\} = 0$, then there is at most $\lfloor m/2 \rfloor$ of $i \in \{1, \dots, m\}$ such that $s_i\{c'_0\} = s_i\{c'_1\} = 1$. Thus, in this case, $\bar{s}[c'_1] \leq \lfloor m/2 \rfloor$. In the other case in which, there is at most $\lfloor m/2 \rfloor$ of $i \in \{1, \dots, m\}$ such that $s_i\{c'_0\} = s_i\{c'_1\} = 0$, then $\bar{s}[c'_0] \leq \lfloor m/2 \rfloor$. And so, in any case, there is a configuration c, for which $\bar{s}[c] \leq \lfloor m/2 \rfloor$. Our choice of $\bar{s}$ was arbitrary, and thus, for any strategy $\bar{s}$, then $\bar{s}[c] \leq \lfloor m/2 \rfloor$. Thus, $\mathrm{HAT}(K_{m,n}, 3) \leq \lfloor m/2 \rfloor$. □

2.5. **An illustrative example.** For the sake of clarity, we will demonstrate the proof above applied to the more concrete case of the hat problem on $K_{4,5}$ and 3 hat colors. To make it even more concrete, let's call these three colors Red, Green, and Blue. Now, the previous theorem gives us that $\mathrm{HAT}(K_{4,5}, 3) = \lfloor \min(4, 5)/2 \rfloor = \lfloor 4/2 \rfloor = 2$. So, to begin, we will first show that there exists a strategy that guarantees two correct answers.

We can find two independent 4-cycles in $K_{4,5}$, and by applying the result of Theorem 2.3.1, each of the two independent 4-cycles has a strategy to guarantee one correct answer. Thus, in total, we can guarantee two correct answers.

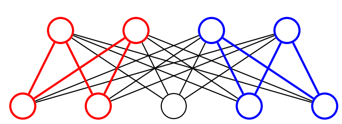
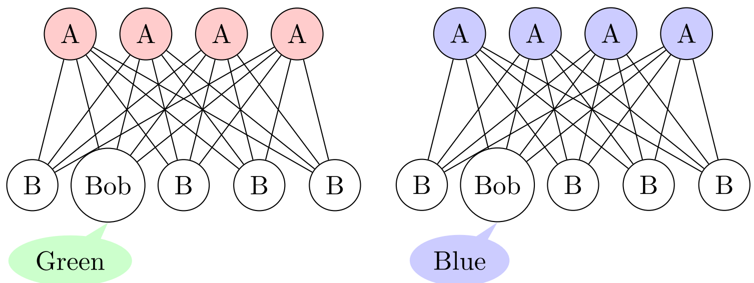


FIGURE 2. Two independent 4-cycles in $K_{4,5}$

Now, we want to prove that no strategy can guarantee more than 2 correct answers. In the proof of Theorem 2.4.1, we show that for any strategy, there is some configuration of hat colors for which there are at most 2 correct answers. Perhaps, this is best understood from the perspective of the mad hatter.

The mad hatter wants there to be as many wrong guesses as possible, so once the participants come up with a strategy, she "tests" this strategy. Let's call the group of 4, Group A, and the group of 5, Group B. The mad hatter knows that Group B can only see Group A and vice versa. So, if she choose that hat colors of Group A, then, she fixes the guesses of Group B. That is, after she chooses the colors of the hats of Group A, then no matter what hat colors are in Group B, Group B's guesses will stay the same.

So, using this knowledge, she first puts Red hats on Group A and takes note of the guesses of Group B. Let's focus on one participant in Group B, Bob. When Bob sees all red in Group A, he guesses "Green." We'll call this the Red Case. This case corresponds to considering c_0 in the proof above. Then the mad hatter puts Blue hats on Group A and takes note of the guesses of Group B. And here, Bob guesses "Blue." We'll call this the Blue Case. This case corresponds to considering c_1 in the proof above.



(A) Red Case (B) Blue Case

FIGURE 3. The Red Case and the Blue Case

In either the Red Case or the Blue Case, Bob does not guess the color Red. Recognizing that Bob does not guess Red corresponds to a_j (more appropriately a_{Bob}) in the above proof. So, the mad hatter decides to put a Red hat on Bob. And just as she can ensure that Bob guesses incorrectly in both the Red Case and the Blue Case, she can ensure that all of group B guesses incorrectly in both cases. This configuration she has found, to ensure that all of Group B is incorrect in both cases, is called c'_0 for the Red Case, and c'_1 in the Blue case in the proof above.

Now, having colored the hats of Group B, the guesses of Group A can no longer change. Suppose in the Red Case, more than 2 of group A guesses "Red." Then more than 2 of Group A is wrong in the Blue Case, as they would guess "Red" also. In other words, at most 1 participant in group A is correct in the Blue Case. Since all of Group B is wrong in either the Red Case or the Blue Case, then at most 1 participant among all the participants is correct. On the other hand, suppose in the Red Case, 2 or less of group A guesses "Red.' Then, at most 2 people in group A is correct in the Red Case. And again, since all of Group B is wrong in either the Red Case or the Blue Case, then at most 2 people among all the participants are correct. Therefore, one of the Red Case or Blue Case causes the strategy to have at most 2 correct answers. And so, no strategy can do better than guaranteeing 2 correct answers.

3. Generalizing the Line of Sages Problem

3.1. **Introduction.** Tanya Khovanova published a paper [1] which explored another type of problem which the paper dubs the "Line of Sages" problem. The setting for this problem varies greatly from the rather general class of hat problems considered in the previous. So, in the Line of Sages problem, we have some number, n people standing in a line each wearing one of $k > n$ hat colors. No two participants will wear the same hat color, thus requiring $k > n$. And the line is formed in such a way that each participant sees, and only sees, the hats of the participants in front of them in the line. That is, participant i in the line can see precisely the hats of participants $i+1, i+2, \dots, n$.

Furthermore, instead of guessing simultaneously, the participants can guess in any order they want. This means that not only do the participants gain information by seeing the hats of other participants, but they also gain information from the guesses of the participants who guess before them. However, once a color is guessed, it may not be guessed again. The goal, as before, is to come up with a strategy which guarantees the most correct guesses. Or, again, less specifically, what is the most correct guesses any strategy can guarantee? In a less formal fashion as the previous section, we notate once more to make it easier to refer to the different variations of this problem.

Notation 3.1.1. *A Line of Sages problem so described is determined by the number of participants n and a number of hat colors $k > n$. Let $\mathrm{SAGE}(n,k)$ be the most correct guesses any strategy can guarantee for the Line of Sages problem with n participants and k hat colors.*

Further, as we did in the previous section: The participants will be referred to as $(1,2,\dots,n)$ where 1 is the backmost participant in the line, and n is the frontmost participant. Also, the hat colors will be $0,1,\dots,k-1$. With this notation, Khovanova found that for any n, $\mathrm{SAGE}(n,n+1) = n-1$.

Because $k > n$, the first guess can never be guaranteed to be correct. Once a strategy is set, since there are more hat colors than participants, there is more than one hat color the first guesser can wear even if the hat colors of the other participants have been set. Thus, for any n and $k > n$, $\mathrm{SAGE}(n,k) \leq n-1$. Another result which has a relatively short proof, is that $\mathrm{SAGE}(2,k) = 1$ for any $k > 2$. Suppose the color of the participant 2's hat is c. Then, participant 1 who sees this hat, can guess $c+1 \pmod{k}$. Then, participant 2 can immediately deduce their hat color from participant 1's guess. This strategy guarantees 1 correct answer, and since, $\mathrm{SAGE}(n,k) \leq n-1$, then we know that $\mathrm{SAGE}(2,k) = 1$.

We conjecture that $\mathrm{SAGE}(n,k) = n-1$ for any n, and $k > n$. This conjecture is supported by Khovanova's result that $\mathrm{SAGE}(n,n+1) = n-1$, the result that the conjecture holds for $n=2$, as well as we prove later in this section, that the conjecture holds for $n=3$. With this conjecture, for a Line of Sages problem with n participants and $k > n$ colors, we call a strategy that guarantees $n-1$ correct guesses a **successful strategy**.

3.2. **Denoting Successful Strategies.** Let's construct a successful strategy for the particular case of 3 participants and 5 colors. We begin by creating participant 1's strategy. Consider the 20 possibilities that participant 1 may see. The first number in the ordered pair represents participant 2's hat color; the second number represents participant 3's hat color.

(1,0)	(2,0)	(3,0)	(4,0)
(0,1)	(2,1)	(3,1)	(4,1)
(0,2)	(1,2)	(3,2)	(4,2)
(0,3)	(1,3)	(2,3)	(4,3)
(0,4)	(1,4)	(2,4)	(3,4)

We want to guarantee that participants 2 and 3 can always call their hat color correctly. So, participant 1 cannot be ambiguous. For example, suppose if participant 1 sees (1,0), then participant 1 calls "4"; also, if participant 1 sees (2,0), then participant 1 calls "4" also. Now, when participant 2 sees that participant 3 has hat color 0 and hears "4", there are still two possibilities for what his own hat color is. We want to avoid this from happening, and thankfully, it is possible. Below is one of many solutions.

(0,1,2)	(0,2,1)	(0,3,4)	(0,4,3)
(1,0,3)	(1,2,4)	(1,3,0)	(1,4,2)
(2,0,4)	(2,1,3)	(2,3,1)	(2,4,0)
(3,0,2)	(3,1,4)	(3,2,0)	(3,4,1)
(4,0,1)	(4,1,0)	(4,2,3)	(4,3,2)

The above table is a condensed way to write each participant's strategy. Let $i \in (1, 2, 3)$. Notice that if we ignore the i-th entry in each ordered triple, then the 20 ordered pairs that remain are the 20 possibilities for what participant i sees and hears. So, based on the ordered pair of what participant i sees and hears, he guesses the i-th entry in the appropriate ordered triple. Such strategies guarantees correct answers from participants 2 and 3.

We can see this strategy in action in an example. Suppose that participant 1 wears hat color 2, participant 2 wears color 1, and participant 3 wears color 0. We can express this color configuration of hat colors shortly as an ordered triple $(2, 1, 0)$. And so, participant 1, sees two colors in front of him: $(1, 0)$. In the table above, there is only one ordered triple for which coordinates 2 and 3 are $(?, 1, 0)$, namely $(4, 1, 0)$. And so, in accordance to the strategy above, participant 1 guesses "4." Now, participant 2 hears the guess 4 from participant 1, and sees that participant 3 is wearing color 0. And again, there is only one ordered triple in the table of the form $(4, ?, 0)$, namely $(4, 1, 0)$, so participant 2 guesses "1." Finally, participant 3 hears two guesses: $(4, 1)$. And again, ther eis only one ordered triple in the table of the form $(4, 1, ?)$, namely $(4, 1, 0)$. So, participant 3 guesses "0." In this example, participants 2 and 3 correctly guess their hat colors.

3.3. **Primary Result.**

Theorem 3.3.1. SAGE$(3, k) = 2$ *for* $k > 3$.

Proof. Consider the problem of 3 people, 5 colors. We would like to make 20 3-tuples such that when the ith element of each tuple is removed, the remaining 2-tuples are distinct. In other words, we would like to fill in the 20 blanks here. Note that if any 2 numbers in the same row are the same, then when the 1st element of each tuple is removed, there will be some identical 2-tuples. Similarly, if any 2 numbers int he same column are the same, then when the 2nd element of each tuple is removed, there will be some identical 2-tuples.

	(1,0,___)	(2,0,___)	(3,0,___)	(4,0,___)
(0,1,___)		(2,1,___)	(3,1,___)	(4,1,___)
(0,2,___)	(1,2,___)		(3,2,___)	(4,2,___)
(0,3,___)	(1,3,___)	(2,3,___)		(4,3,___)
(0,4,___)	(1,4,___)	(2,4,___)	(3,4,___)	

Seeing a form like this suggests converting this into a Latin square that we have to fill in:

0				
	1			
		2		
			3	
				4

As an example, look at the cell in the top right. Either a 1, 2, or 3 can be placed in that cell in the strategy table, as well as in the Latin square. This equivalence between strategies and Latin squares generalizes. That is, the problem of filling in a k by k Latin square, where the diagonal is already filled as above, is equivalent to the Line of Sages problem for 3 people and k colors.

For odd k, the Latin square has a linear construction. If we create a "coordinate axis" around the Latin square, then we can fill in the cells by the function $f(x, y) \equiv 2x - y \pmod{k}$. The case for $k = 5$ colors is shown below.

	x	0	1	2	3	4
y						
0		0	2	4	1	3
1		4	1	3	0	2
2		3	0	2	4	1
3		2	4	1	3	0
4		1	3	0	2	4

For even k, the construction is not linear, but is based off the odd k construction. We take the Latin square of order $k-1$, where $f(x, y) = 2x - y$ as before. We will take $k = 6$ as an example. First, we will show that the colored numbers are an arrangement of $0, 1, \cdots, k-1$.

0	2	4	1	3	
4	1	3	0	2	
3	0	2	4	1	
2	4	1	3	0	
1	3	0	2	4	

The colored numbers are in cells of the form $(i, i+1 \pmod{k})$, where $i \in \{0, 1, \cdots, k-1\}$. The number in the cell is

$$f(i, i+1 \pmod{k}) \equiv 2i - (i+1) \equiv i - 1 \pmod{k}$$

As i ranges from 0 to $k-1$, $i-1 \pmod{k}$ also ranges from 0 to $k-1$. Therefore the colored numbers are an arrangement of $0, 1, \cdots, k-1$. What this means is that we can now copy these colored numbers into a new row and column, and that new row and column will still be Latin:

0	2	4	1	3	2
4	1	3	0	2	3
3	0	2	4	1	4
2	4	1	3	0	0
1	3	0	2	4	1
1	2	3	4	0	

Finally, we replace the original colored numbers with k's (in this case, 5's) and also put a k in the lower right corner, which finishes the construction. Thus, we have shown that there exists a k by k Latin square where the diagonal is filled in increasing order.

0	5	4	1	3	2
4	1	5	0	2	3
3	0	2	5	1	4
2	4	1	3	5	0
5	3	0	2	4	1
1	2	3	4	0	5

Since, such Latin squares can be converted to successful strategies for 3 participants and k colors, then $\text{SAGE}(3, k) \geq 2$. Also, we know $\text{SAGE}(3, k) \leq 2$, because the adversary can always force the first person in line to be incorrect. Thus, we have shown that $\text{SAGE}(3, k) = 2$, as desired. □

3.4. **Further Results.** So far we've constructed successful strategies using Latin squares. We can also use Steiner systems to construct successful strategies. Let's consider $\text{SAGE}(4, 8)$ as an example. The Steiner system $S(3, 4, 8)$ is a family of subsets of $\{1, 2, \ldots, 8\}$ of size 4 ("blocks"). Every possible subset of size 3 is in exactly one block:

(1,2,4,8)	(3,5,6,7)
(2,3,5,8)	(1,4,6,7)
(3,4,6,8)	(1,2,5,7)
(4,5,7,8)	(1,2,3,6)
(1,5,6,8)	(2,3,4,7)
(2,6,7,8)	(1,3,4,5)
(1,3,7,8)	(2,4,5,6)

Now to construct our successful strategy, for each of the above 14 blocks, we take every permutation of it. This gives us $14 * 24 = 336$ tuples.

For each participant, they will hear/see an ordered 3-tuple. As a property of the Steiner system, the unordered set of 3 elements in this 3-tuple is in exactly one of the 14 blocks. There is exactly one permutation of that block which has the elements of the 3-tuple in the correct order.

As an example, suppose participant 2 hears participant 1 say "5", and sees participants 3 and 4 wearing colors 6 and 7, respectively. So, we must find a tuple of the form $(5, ?, 6, 7)$. The only block containing the set $(5, 6, 7)$ is $(3, 5, 6, 7)$. So, the only permutation among the 336 tuples that matches is $(5, 3, 6, 7)$. Therefore participant 2 should say "3".

In general, a Steiner system $S(n-1, n, k)$ can be transformed into a successful strategy for $\text{SAGE}(n, k)$. By Keevash's result on Steiner systems [4], for every n, there are infinitely many k such that $S(n-1, n, k)$ exists. Alas, infinitely many k doesn't mean all k.

References

[1] T. Khovanova, "A line of sages," [Online]. Available: http://www.tanyakhovanova.com/publications/ALineOfWizards.pdf.

[2] S. Butler, M. T. Hajiaghayi, R. D. Kleinberg, and T. Leighton, *Hat guessing game*. [Online]. Available: https://www.cs.cornell.edu/~rdk/papers/hats.pdf.

[3] W. Szczechla, "The three colour hat guessing game on cycle graphs," *The Electronic Journal of Combinatorics*, vol. 24, no. 1, 2017. [Online]. Available: http://www.combinatorics.org/ojs/index.php/eljc/article/view/v24i1p37/pdf.

[4] P. Keevash, "The existence of designs," [Online]. Available: https://people.maths.ox.ac.uk/keevash/papers/designs.pdf.